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# AN ANALYTICAL SOLUTION FOR SOLIDIFICATION OF A MOVING WARM LIQUID ONTO AN ISOTHERMAL COLD WALL

JOSEPH M. SAVINO and ROBERT SIEGEL

Lewis Research Center, National Aeronautics and Space Administration, Cleveland, Ohio

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**Abstract**—Successive solutions for the instantaneous frozen layer thickness and temperature profile in the solidified layer were generated by an analytical iteration technique. Each iteration was more accurate than the preceding one and the succession of solutions converged rapidly. Analytical expressions obtained were of a simple form and agreed with numerical and approximate solutions of other investigators.

## NOMENCLATURE

$c_p$, specific heat;
$G$, integral defined in equation (9);
$h$, convective heat transfer coefficient;
$I$, integrals defined in equation (8);
$k$, thermal conductivity of solidified material;
$L$, latent heat of fusion;
$S$, dimensionless parameter, $c_p(t_f - t_w)/L$;
$t$, temperature;
$T$, dimensionless temperature, $(t - t_f)/(t_w - t_f)$;
$x$, position coordinate in frozen layer measured from wall;
$X$, dimensionless coordinate, $x/\delta_s$;
$\Delta$, dimensionless frozen layer thickness, $\delta/\delta_s$;
$\delta$, thickness of frozen layer;
$\delta_s$, thickness of frozen layer at steady state;
$\Theta$, dimensionless time, $\theta h_l(t_l - t_f)/\rho L \delta_s$;
$\theta$, time from start of solidification;
$\xi$, dimensionless coordinate, $x/\delta$;
$\rho$, density.

Subscripts

$f$, at freezing temperature;
$l$, liquid phase of solidifying material;
$s$, steady state;
$w$, wall;
I, II, III, IV, successive iterative approximations.

## INTRODUCTION

AN ANALYTICAL solution is presented for the type of moving boundary transient heat conduction problem encountered in the freezing of a liquid. Finding a solution for the frozen layer thickness variation with time is usually very difficult because of the nonlinear mathematical complexities introduced by the presence of the moving interface. For this reason only a few exact analytical solutions have been found, Carslaw and Jaeger [1]. Because of the many practical situations involving melting and freezing, the search continues for analytical solutions of a reasonably simple form.

This work is a continuation of the studies we reported in [2–4]. In those references an analytical method was developed and applied to the general problem of a warm flowing liquid freezing onto a plane wall that is convectively cooled on the opposite side. A relation for the frozen layer thickness as a function of time was found in the form of an integral equation that was solved by analytical iterations. A limiting case of the general solution is when the frozen layer is formed on a wall at constant temperature. This case is considered here, and it was possible to carry out four iterations as closed form analytical expressions. The third and fourth iterations converged to within 1 percent of each other so that they closely represent the exact solution.

Results on this problem have been published only in recent years. Libby and Chen [5] presented a numerical solution and some approximate analytical solutions that are valid near the beginning and end of the growth period. Lapadula and Mueller [6] developed a simple approximate analytical solution by use of a variational method. Beaubouef and Chapman [7] obtained some numerical solutions. The results of these references are used for comparison with those derived in this paper.

## ANALYSIS

The model analyzed is shown in Fig. 1. A liquid at a fixed bulk temperature $t_l$ above its freezing point $t_f$ flows over a cold wall held at constant temperature $t_w < t_f$ for $\theta > 0$. A frozen layer forms on the wall and grows to a steady thickness $\delta_s$. The liquid-solid interface is taken to always be at $t_f$ as was experimentally demonstrated in [3]. It is assumed that the frozen layer properties and the convective heat transfer coefficient remain constant with time.

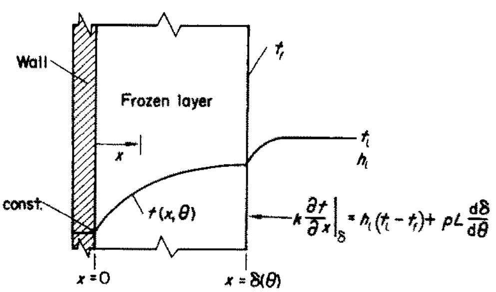


FIG. 1. Frozen layer formation on wall at constant temperature.

An important and useful fact is that the flowing liquid supplies a constant convective heat flux $h_l(t_l - t_f)$ to the frozen interface at all times after the frozen layer is formed. From a heat balance at steady state $h_l(t_l - t_f) = k(t_f - t_w)/\delta_s$, the thickness $\delta_s$ is calculated which is used as a reference length

$$\delta_s = \frac{k(t_f - t_w)}{h_l(t_l - t_f)}. \tag{1}$$

To determine the frozen layer growth and temperature distribution, begin with the transient one-dimensional heat conduction equation

$$k\frac{\partial^2 t}{\partial x^2} = \rho c_p \frac{\partial t}{\partial \theta} \tag{2}$$

and integrate from $x$ to $\delta$:

$$k\frac{\partial t}{\partial x}\bigg|_\delta - k\frac{\partial t}{\partial x}\bigg|_x = \rho c_p \int\limits_x^\delta \frac{\partial t}{\partial \theta}\,\mathrm{d}x. \tag{3}$$

Substitute the boundary condition at $x = \delta$ which is

$$k\frac{\partial t}{\partial x}\bigg|_\delta = h_l(t_l - t_f) + \rho L\frac{\mathrm{d}\delta}{\mathrm{d}\theta} \tag{4}$$

and equation (3) becomes

$$k\frac{\partial t}{\partial x}\bigg|_x = \rho L\frac{\mathrm{d}\delta}{\mathrm{d}\theta} + h_l(t_l - t_f) - \rho c_p \int\limits_x^\delta \frac{\partial t}{\partial \theta}\,\mathrm{d}x. \tag{5}$$

Integrating equation (5) from $x = 0$ to $x$ and rearranging provides an expression for the temperature distribution,

$$t(x,\theta) = t_w + \frac{\rho L}{k}\frac{\mathrm{d}\delta}{\mathrm{d}\theta}x + \frac{h_l}{k}(t_l - t_f)x - \frac{\rho c_p}{k}\int\limits_0^x\int\limits_x^\delta \frac{\partial t}{\partial \theta}\,\mathrm{d}x\mathrm{d}x \tag{6}$$

which can be placed in dimensionless form

$$T = 1 - X - X\frac{\mathrm{d}\Delta}{\mathrm{d}\Theta} - S\int\limits_0^X\int\limits_X^\Delta \frac{\partial T}{\partial \Theta}\,\mathrm{d}X\mathrm{d}X. \tag{7}$$

By applying the rules for differentiating an integral, as shown in [2], to remove the derivative from under the integral sign and using the relation $\partial I(X, \Theta)/\partial \Theta = (\mathrm{d}\Delta/\mathrm{d}\Theta)[\partial I(X, \Delta)/\partial \Delta]$ equation (7) can be written as

$$T = 1 - X - \left(X + S\frac{\partial I}{\partial \Delta}\right)\frac{\mathrm{d}\Delta}{\mathrm{d}\Theta}, \qquad (8)$$

where

$$I(X, \Delta) \equiv \int_0^X XT\,\mathrm{d}X + X\int_X^\Delta T\,\mathrm{d}X.$$

By letting $X = \Delta$ in equation (8) and noting that $T(\Delta) = 0$ the equation for the growth rate is obtained,

$$\frac{\mathrm{d}\Delta}{\mathrm{d}\Theta} = \frac{1 - \Delta}{\Delta + S(\mathrm{d}G/\mathrm{d}\Delta)} \qquad (9)$$

where

$$G(\Delta) \equiv I(X = \Delta, \Delta) = \int_0^\Delta XT\,\mathrm{d}X.$$

Integrating equation (9) using the initial condition $\Delta = 0$ when $\Theta = 0$ results in

$$\Theta = -\Delta - \ln(1 - \Delta) + S\int_0^\Delta \frac{\mathrm{d}G/\mathrm{d}\Delta}{1 - \Delta}\mathrm{d}\Delta. \qquad (10)$$

Equation (8) is now changed by eliminating $\mathrm{d}\Delta/\mathrm{d}\Theta$ with equation (9) to obtain

$$T = 1 - X - (1 - \Delta)\left\{\frac{X + S(\partial I/\partial \Delta)}{\Delta + S(\mathrm{d}G/\mathrm{d}\Delta)}\right\}. \qquad (11)$$

Equation (11) is a complicated integral equation for the temperature distribution in the frozen layer. If it could be solved for $T(X, \Delta)$, then $G(\Delta)$ could be evaluated and the integral of equation (10) found to relate $\Delta$ and $\Theta$ explictly.

Equations (10) and (11) are now solved by an analytical iteration procedure. A first order approximation for $\Theta(\Delta)$ and $T(X, \Delta)$ is obtained by neglecting the heat capacity, that is by letting $S = 0$ in equations (10) and (11)

$$\Theta_{\mathrm{I}} = -\Delta - \ln(1 - \Delta) \qquad (12a)$$

$$T_{\mathrm{I}} = 1 - \frac{X}{\Delta} = 1 - \xi. \qquad (12b)$$

Then equation (12b) is used to derive $I_{\mathrm{I}}(X, \Delta)$ and $G_{\mathrm{I}}(\Delta)$ for use in equations (10) and (11) to obtain the second analytical iteration $\Theta_{\mathrm{II}}(\Delta)$ and $T_{\mathrm{II}}(X, \Delta)$:

$$\Theta_{\mathrm{II}} = \left(1 + \frac{S}{3}\right)[-\Delta - \ln(1 - \Delta)] \qquad (13a)$$

$$T_{\mathrm{II}} = 1 - \xi\Delta - \frac{3}{3 + S}\left[\left(1 + \frac{S}{2}\right)\xi - \frac{S}{6}\xi^3\right] \cdot (1 - \Delta). \qquad (13b)$$

This procedure is repeated. Each successive iteration is used to generate a more exact approximation until a converged solution is obtained or until the equations become too unwieldly to manipulate.

For this problem we were able to generate four successive iterations. Having outlined the method used, the third and fourth iterations are given in their final form without taking space for the intermediate mathematical manipulations.

$$\Theta_{\mathrm{III}} = \frac{3 + 2S + \frac{2}{5}S^2}{3 + S}[-\Delta - \ln(1 - \Delta)] - \frac{S^2}{3 + S}\frac{\Delta^2}{10} \qquad (14a)$$

$$T_{\mathrm{III}} = 1 - \xi\Delta - \frac{(1 - \Delta)(3 + S)}{\Delta\left[3 + 2S + \frac{S^2}{5}(1 + \Delta)\right]}\left\{\xi\ [1 + S(1 - \Delta)] + \frac{S}{2(3 + S)}\left[\xi\Delta\left(-3 + 6\Delta - \frac{5S}{4} + \frac{5S\Delta}{2}\right) - \xi^3\Delta\left(1 + \frac{S}{2}\right) + \xi^5\Delta^2\left(\frac{3S}{20\Delta} - \frac{S}{10}\right)\right]\right\} \qquad (14b)$$

$$\Theta_{IV} = \Theta_I + \frac{13}{42}S\Delta^2 + \frac{1}{7}\left(\frac{5}{S} + 1 - 3S\right)\Delta - \frac{S}{3 + 2S + (S^2/5)(1+\Delta)}\left[\left(1 + \frac{11}{15}S + \frac{3}{35}S^2\right)\Delta^2 + \frac{8}{105}S^2\Delta^3\right] + \left[-\frac{4}{21}S + \frac{5}{21} - \frac{5}{14S} - \frac{65}{14S^2} - \frac{75}{14S^3}\right]$$

$$\cdot \ln\left[\frac{-S^2\Delta^2 - 5(3+2S)\Delta + (15 + 10S + S^2)}{S^2 + 10S + 15}\right] - \left(\frac{3}{35}S^3 + \frac{47}{105}S^2 + \frac{16}{21}S - \frac{13}{7} - \frac{25}{2S} - \frac{345}{14S^2} - \frac{225}{14S^3}\right) \tag{15a}$$

$$\cdot \frac{1}{\sqrt{M}} \ln \frac{\left|\dfrac{\frac{2}{5}S^2\Delta + 3 + 2S - \sqrt{M}}{\frac{2}{5}S^2\Delta + 3 + 2S + \sqrt{M}}\right|}{\left|\dfrac{3 + 2S - \sqrt{M}}{3 + 2S + \sqrt{M}}\right|}$$

where $M \equiv 9 + 12S + \frac{32}{5}S^2 + \frac{8}{5}S^3 + \frac{4}{25}S^4$

$$T_{IV}(X, \Delta) = 1 - \xi\Delta - (1 - \Delta)\left(\frac{\xi\Delta + S\dfrac{\partial I_{III}}{\partial \Delta}}{\Delta + S\dfrac{dG_{III}}{d\Delta}}\right) \tag{15b}$$

where

$$\frac{dG_{III}}{d\Delta} = \Delta - \Delta^2 - \frac{1-\Delta}{3 + 2S + (S^2/5)(1+\Delta)}\left[2\Delta\left(1 + \frac{11}{15}S + \frac{3}{35}S^2\right) + \frac{8}{35}S^2\Delta^2\right] + \left\{\frac{3 + 2S + (2S^2/5)}{[3 + 2S + (S^2/5)(1+\Delta)]^2}\right\}\left[\Delta^2\left(1 + \frac{11}{15}S + \frac{3}{35}S^2\right) + \frac{8}{105}S^2\Delta^3\right]$$

$$\frac{\partial I_{III}}{\partial \Delta} = (1-\Delta)\xi\Delta - \frac{(1-\Delta)}{2[3 + 2S + (S^2/5)(1+\Delta)]}\left[\xi\Delta\left(3 + \frac{9}{4}S + \frac{11}{40}S^2 + \frac{7}{15}S^2\Delta\right) + \xi^3\Delta\left(1 + \frac{5S}{6} + \frac{S^2}{8}\right) - \xi^5\Delta\left(\frac{3}{20}S + \frac{3}{40}S^2\right) + \xi^7\Delta^2\left(\frac{S^2}{56\Delta} - \frac{S^2}{105}\right)\right]$$

$$+ \left\{\frac{3 + 2S + \frac{2}{5}S^2}{2[3 + 2S + (S^2/5)(1+\Delta)]^2}\right\}\left\{\xi\Delta\left[\Delta\left(3 + \frac{9}{4}S + \frac{11}{40}S^2\right) + \frac{7}{30}S^2\Delta^2\right] - \xi^3\Delta^3\left[\frac{1}{\Delta}\left(1 + \frac{5}{6}S + \frac{S^2}{8}\right) + \frac{S^2}{12}\right] + \xi^5\Delta^2\left[\frac{S}{20} + \frac{S^2}{40}\right] + \xi^7\Delta^3\left[-\frac{S^2}{280\Delta} + \frac{S^2}{420}\right]\right\}.$$

As will be shown in the following section, the $\Theta_{IV}(\Delta)$ and $T_{IV}$ are converged, and it is unnecessary to carry out additional iterations.

## RESULTS AND DISCUSSION

*Comparison of solutions*

Here the successive iterations are compared with each other, and the converged solution is compared with the results of other investigators. In Fig. 2 are shown the temperature distributions in the frozen layer for various values of $\Delta$ and $S = 5$ which provides a very large heat capacity effect. The $T_{III}$ and $T_{IV}$ curves are very close to each other; their calculated values deviate only slightly in the third significant figure.

Similarly, Fig. 3 displays the relation between $\Delta$ and $\Theta$ for values of the subcooling parameter $S = 0$, 1·0, 3·0, and 5·0. As with the temperature distribution, $\Theta_{III}$ and $\Theta_{IV}$ are extremely close to each other, and can be considered converged. The simplicity in the forms for $\Theta_{III}$ and $T_{III}$, equations (14a) and (b), makes them quite useful for practical applications. Because the simple second iterative solution deviates at most by 2 per cent from the converged solution, it is also useful for practical problems.

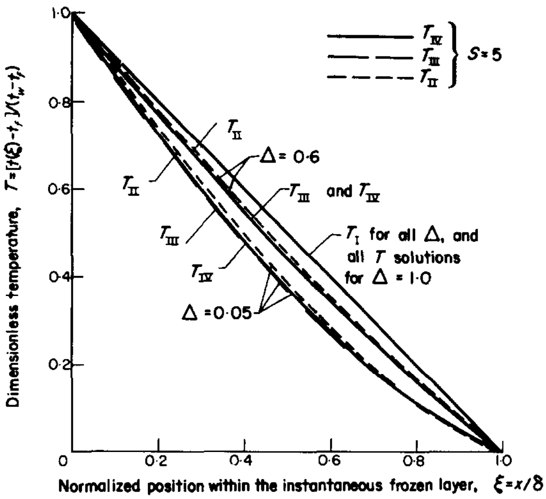


FIG. 2. Comparison of temperature distributions in frozen layer computed by each successive approximate solution for two fractional thicknesses, $S = 5$.

A comparison is made in Fig. 4 with the results of Libby and Chen [5], Lapadula and

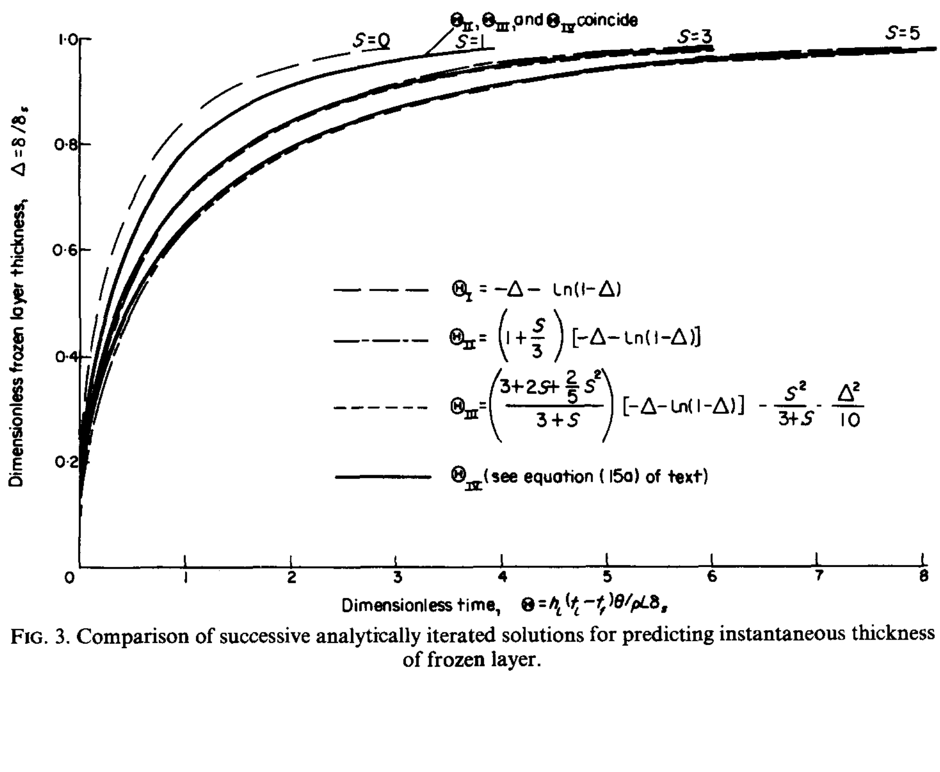


FIG. 3. Comparison of successive analytically iterated solutions for predicting instantaneous thickness of frozen layer.

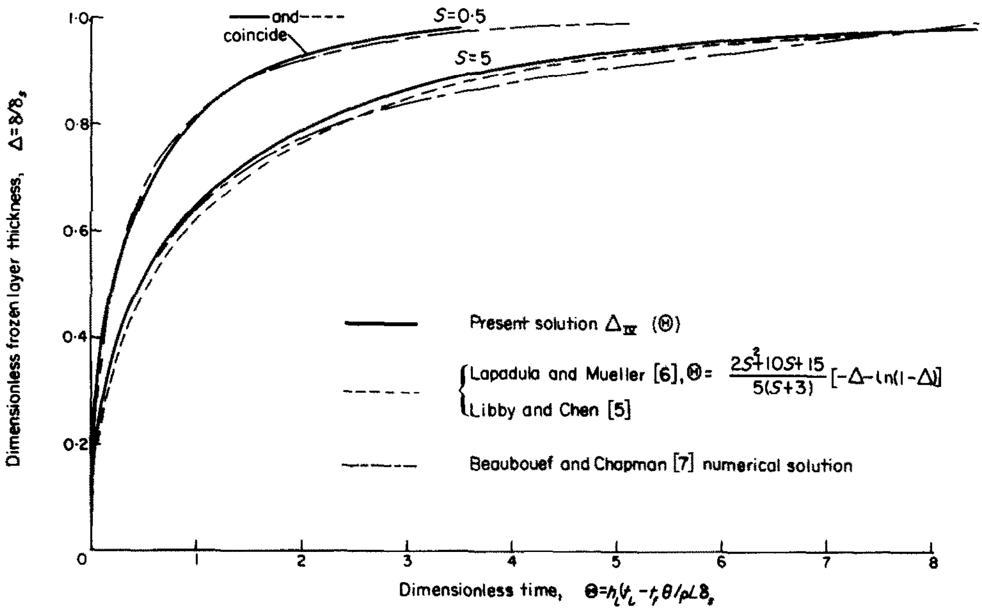


FIG. 4. Comparison of present converged solution with results of other investigators.

Mueller [6] and Beaubouef and Chapman [7]. Clearly, there is good agreement of our converged solution with both the numerical and analytical approximate solutions. This agreement indicates that the iterations given here have converged to the correct solution.

## CONCLUSIONS

A method for generating solutions by analytical iterations was applied for predicting the instantaneous thickness of a frozen layer that forms on a cold wall at constant temperature in a stream of liquid. The important feature of the method is that the successive approximate solutions for the temperature distribution and instantaneous thickness quickly converged. As a result approximate solutions of a very simple analytical form were found that agreed within 2 per cent with the more complex converged solution.

## REFERENCES


1. H. S. CARSLAW and J. C. JAEGER, *Conduction of Heat in Solids*. Clarendon Press, Oxford (1959).
2. R. SIEGEL and J. M. SAVINO, An analysis of the transient solidification of a flowing warm liquid on a convectively cooled wall. *Proc. 3rd Int. Heat Transfer Conf.* Vol. 4, pp. 141–151. Am. Soc. Mech. Engrs., New York (1966).
3. J. M. SAVINO and R. SIEGEL, Experimental and analytical study of the transient solidification of a warm liquid flowing over a chilled flat plate. NASA TN D-4015 (1967).
4. R. SIEGEL and J. M. SAVINO, Transient solidification of a flowing liquid on a cold plate including heat capacities of frozen layer and plate. NASA TN D-4353 (1968); Paper No. 67-WA/HT-34 ASME Winter Annual Meeting, Pittsburg, Pa, Nov. 17, 1967.
5. P. A. LIBBY and S. CHEN, The growth of a deposited layer on a cold surface. *Int. J. Heat Mass Transfer* **8**, 395–402 (1965).
6. C. LAPADULA and W. K. MUELLER, Heat conduction with solidification and a convective boundary condition at the freezing front. *Int. J. Heat Mass Transfer* **9**. 702–705 (1966).
7. R. T. BEAUBOUEF and A. J. CHAPMAN, Freezing of fluids in forced flow. *Int. J. Heat Mass Transfer* **10**, 1581–1588 (1967).


**Résumé**—Des solutions successives pour l'épaisseur instantanée de la couche gelée et le profil de vitesse dans la couche solidifiée fournies par une technique d'iteration analytique. Chaque itération était plus précise que la précédente et la succession des solutions convergeait rapidement. Les expressions analytiques obtenues étaient de forme simple et en accord avec les solutions numériques et approchées d'autres chercheurs.

**Zusammenfassung**—Lösungsfolgen für die augenblickliche Dicke einer gefrorenen Schicht und Temperaturprofile in der verfestigten Schicht liessen sich nach einer analytischen Iterationstechnik angeben. Jede Iteration war genauer als die vorhergehende und die Lüsungsfolgen konvergierten sehr rasch. Dier erhaltenen Ausdrücke waren von einfacher Form und stimmten überein mit numerischen und angenäherten Lösungen anderer Autoren.

**Аннотация**—С помощью метода итерации получены последовательные решения для мгновенной толщины замерзшего слоя и температурного профиля в отвердевшем слое. Каждая последующая итерация производилась тщательнее предыдущей, и последовательность решений быстро сходилась. Получены простые по форме аналитические выражения, которые согласуются с численными и приближенными решениями других исследователей.